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Two-Pion-Exchange Contribution to the Λ-Nucleon Scattering Matrix*†

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The two-pion-exchange (TPE) contribution to the Λ-nucleon scattering matrix has been calculated with the Dyson S-matrix formalism under the assumption of even $\Lambda-\Sigma$ parity and a universal pion-baryon interaction. This calculation leads to a TPE contribution to the zero-energy Λ-nucleon scattering length which is spin-independent. For scattering at finite energies, the scattering matrix is spin-dependent and contains an anti-symmetric spin-orbit term, which does not appear in the charge-symmetric nucleon-nucleon scattering matrix. The results suggest the use of a semiphenomenological Λ-nucleon potential which, for large separations, is predominantly spin-independent.

I. INTRODUCTION

ANALYSES of the hypernuclear binding energy data in terms of phenomenological two-body Λ-nucleon potentials have led to a specification of parameters characterizing the Λ-nucleon interaction in S states for assumed interaction ranges corresponding to the lowest order pion- and kaon-exchange mechanisms which can give rise to a charge-independent interaction.[1] The two-body Λ-nucleon interaction potentials which have been thus deduced are strong and highly spin-dependent.[1] Since the discovery of this spin dependence by Dalitz,[2] there have been several attempts to reproduce it in terms of meson-theoretic potentials.[3–7] The object of these calculations has been to determine the extent to which the various meson-exchange mechanisms considered can lead to agreement with the strength and spin dependence of the phenomenological Λ-nucleon interaction potentials for various possible forms of the meson-baryon interaction with appropriately chosen values of the relevant coupling constants. Attempts to reproduce the features of the phenomenological Λ-nucleon interaction in terms of a dominant kaon-exchange mechanism have been discouraging.[5,6] In the most recent calculations, attention has been focused on simple pion-exchange mechanisms.[7]

The lowest order pion-exchange mechanisms which can contribute to a charge-independent Λ-nucleon interaction are those in which two pions are exchanged between the two baryons. Attempts to evaluate the two-pion-exchange (TPE) contribution to the Λ-nucleon interaction have been made in terms of Tamm-Dancoff potentials[8] corresponding to the no-pair diagrams of Fig. 1.[9] The contribution of the improper diagram of Fig. 1(d) can either be calculated directly[3,10] or included as an iteration of the effect of the second-order diagram of Fig. 1(a). In the latter procedure, which was introduced by Lichtenberg and Ross,[4] the Λ- and Σ-nucleon systems are described by the coupled Schrödinger equations,

$$-\frac{\hbar^2}{2\mu_\Lambda}\nabla^2\psi_\Lambda+(V_\Lambda-E_\Lambda)\psi_\Lambda=-V_{\Lambda\Sigma}\psi_\Sigma, \tag{1a}$$

$$-\frac{\hbar^2}{2\mu_\Sigma}\nabla^2\psi_\Sigma+(V_\Sigma-E_\Sigma)\psi_\Sigma=-V_{\Sigma\Lambda}\psi_\Lambda. \tag{1b}$$

The wave functions ψ_Λ and ψ_Σ describe the Λ- and Σ-nucleon systems with energies E_Λ and E_Σ and reduced masses μ_Λ and μ_Σ, respectively. The potential V_Λ is the Tamm-Dancoff potential which, through fourth order in the pion-baryon coupling constants, corresponds to the proper diagrams of Figs. 1(b) and 1(c), when the contributions of diagrams with intermediate baryon pairs are neglected. Similarly, the second- and fourth-order contributions to V_Σ, $V_{\Lambda\Sigma}$, and $V_{\Sigma\Lambda}$ correspond to the proper diagrams of Figs. 1(a), 1(b), and (1c).

The zero-energy Λ-nucleon scattering lengths in the singlet and triplet spin states have been calculated by numerical integration of Eqs. (1).[4,7] The second- and

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† In this paper, the designation TPE is restricted to two-pion-exchange processes of fourth order in the pion-baryon coupling constants.

[1] For the results of analyses in terms of central potentials without hard cores see, for example, R. H. Dalitz and B. W. Downs, Phys. Rev. **111**, 967 (1958); and B. W. Downs and R. H. Dalitz, *ibid.* **114**, 593 (1959). For analyses in terms of central potentials with hard cores see, for example, K. Dietrich, R. Folk, and H. J. Mang, in *Proceedings of the Rutherford Jubilee Conference, Manchester, 1961*, edited by J. B. Birks (Heywood and Company Ltd., London, 1961), p. 165, and B. W. Downs, D. R. Smith, and T. N. Truong, Phys. Rev. **129**, 2730 (1963).

[2] R. H. Dalitz, in *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), p. 40.

[3] D. B. Lichtenberg and M. Ross, Phys. Rev. **103**, 1131 (L) (1956).

[4] D. B. Lichtenberg and M. Ross, Phys. Rev. **107**, 1714 (1957).

[5] D. B. Lichtenberg and M. Ross, Phys. Rev. **109**, 2163 (1958).

[6] F. Ferrari and L. Fonda, Nuovo Cimento **9**, 842 (1958).

[7] J. J. de Swart and C. K. Iddings, Phys. Rev. **128**, 2810 (1962).

[8] The term "Tamm-Dancoff potential" is used here to designate any Tamm-Dancoff-like field-theoretic potential intended to be used in a Schrödinger equation.

[9] Although the diagram of Fig. 1(d) does not contribute to the nucleon-nucleon Tamm-Dancoff potential [being an iteration of the contribution of the diagram of Fig. 1(a)], it does contribute to the Λ-nucleon Tamm-Dancoff potential when the interaction Hamiltonian which turns a Λ into a Σ and vice versa is introduced. See reference 3.

[10] See M. J. Moravcsik and H. P. Noyes, Ann. Rev. Nucl. Sci. **11**, 95 (1961) for a summary of the results of calculations of static nucleon-nucleon potentials corresponding to the diagrams of Fig. 1.

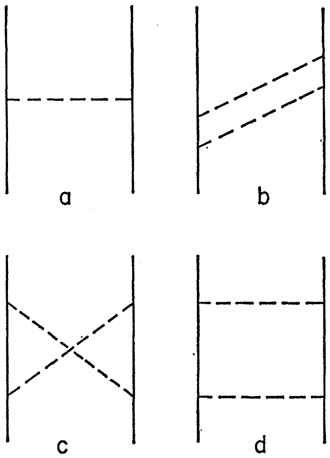

FIG. 1. No-pair pion-exchange diagrams which contribute to the hyperon-nucleon Tamm-Dancoff potentials V_Λ, V_Σ, $V_{\Lambda\Sigma}$, and $V_{\Sigma\Lambda}$ through fourth-order in the pion-baryon coupling constants. One vertical line of each diagram is a nucleon line; the other, a hyperon (Λ and/or Σ) line. Figures 1(a), 1(b), and 1(c) are proper diagrams in which there is at least one pion in every intermediate state; Fig. 1(d) is an improper diagram in which there is an intermediate state containing no pions.

fourth-order potentials described in the preceding paragraph have been assumed to give a good representation of the hyperon-nucleon interactions for large separations; in the region of small separations, the interactions have been represented by hard-core potentials.[4,7] The validity of this procedure can be investigated with the help of a formal solution of the coupled Eqs. (1) in which the Σ-nucleon wave function ψ_Σ is eliminated. That is,

$$\left(-\frac{\hbar^2}{2\mu_\Lambda}\nabla^2-E_\Lambda\right)\psi_\Lambda = -\{V_\Lambda - V_{\Lambda\Sigma}(1+G_\Sigma V_\Sigma)^{-1}G_\Sigma V_{\Sigma\Lambda}\}\psi_\Lambda, \quad (2)$$

where

$$G_\Sigma=\left(-\frac{\hbar^2}{2\mu_\Sigma}\nabla^2-E_\Sigma\right)^{-1} \quad (3)$$

is the Green's function for the free Σ-nucleon system. The { } factor on the right-hand side of (2) is the (nonlocal) effective Λ-nucleon potential $V_{\Lambda-N}$. The leading terms in an expansion of this potential are

$$V_{\Lambda-N}=V_\Lambda - V_{\Lambda\Sigma}G_\Sigma V_{\Sigma\Lambda} + V_{\Lambda\Sigma}G_\Sigma V_\Sigma G_\Sigma V_{\Sigma\Lambda} + \cdots. \quad (4)$$

When the potentials V_Λ, V_Σ, $V_{\Lambda\Sigma}$, and $V_{\Sigma\Lambda}$ are evaluated through fourth order in the pion-baryon coupling constants, the first term in (4) corresponds to the fourth-order proper diagrams of Figs. 1(b) and 1(c). The second term contains contributions of fourth, sixth, and eighth order; and the third term contains contributions of sixth through twelfth order. In addition to the higher-than-fourth-order contributions contained in (4), others are introduced because the solution of Eq. (2) involves an iteration of the effective potential. The sixth-order contributions to the second and third terms in (4) correspond to improper diagrams of the form of Figs. 2(a) and 2(b), respectively. When the potentials appearing in Eqs. (1) are evaluated only through fourth order in the pion-baryon coupling constants, however, contributions to the effective potential (4) corresponding to proper sixth-order diagrams of the form of Fig. 2(c) are not included. A similar remark applies to contributions of still higher order.[11] When hard cores are included in the hyperon-nucleon potentials, some of the higher order contributions to these potentials are important only inside the core region where they are dominated by the core. It is important to know which higher order contributions are eliminated by the introduction of the hard cores. The results of recent calculations by de Swart and Iddings[7] indicate that sixth-order contributions (at least) are not eliminated by hard cores of radii about $0.35(\hbar/M_\pi c)$.[12] It would, therefore, seem that a self-consistent calculation based on Eqs. (1), in which sixth-order effects are important, should include consideration of the contributions of proper sixth-order diagrams.[13]

On account of the extreme difficulty of evaluating the contributions to a Tamm-Dancoff potential corresponding to sixth-order diagrams, an attempt to make a self-consistent calculation based on Eqs. (1), including sixth-order effects, would appear to be unpromising. A

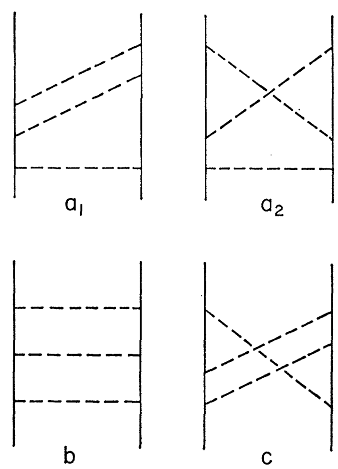

FIG. 2. Some sixth-order pion-exchange diagrams which contribute to the effective Λ-nucleon potential (4). The only completely proper diagram is that of Fig. 2(c).

[11] That the use of the coupled Eqs. (1) introduces contributions to the effective potential corresponding to improper diagrams not explicitly considered was stated by Lichtenberg and Ross in reference 4, in which they introduced this technique.

[12] In reference 7, de Swart and Iddings considered the case of even $\Lambda-\Sigma$ parity and a universal pion-baryon interaction with the possibility of different coupling constants for the Yukawa interactions (a) $N\leftrightarrow N+\pi$, (b) $\Sigma\leftrightarrow\Sigma+\pi$, and (c) $\Lambda\leftrightarrow\Sigma+\pi$. Having evaluated the potentials appearing in Eqs. (1) through fourth order in the pion-baryon coupling constants, they were able to reproduce highly spin-dependent zero-energy Λ-nucleon scattering lengths with (1) essentially the same coupling constants for the interactions (a), (b), and (c) and the same hard-core radius $[0.35(\hbar/M_\pi c)]$ in singlet and triplet spin states, and with (2) the same coupling constants for the interactions (a) and (c) and a zero coupling constant for the interaction (b) with hard-core radii of $[0.31(\hbar/M_\pi c)]$ and $[0.39(\hbar/M_\pi c)]$ in the singlet and triplet spin states, respectively. Sixth-order contributions corresponding to the improper diagrams of Figs. 2(a) and 2(b) are included in situation (1) but not in situation (2).

[13] J. J. de Swart and C. Dullemond, Ann. Phys. (N. Y.) **16**, 263 (1961) have used the coupled Eqs. (1) to calculate hyperon-nucleon scattering parameters in terms of phenomenological hyperon-nucleon potentials. These calculations are presumably completely self-consistent.

self-consistent calculation of the fourth-order (TPE) contributions to the Λ-nucleon scattering amplitude could, however, in principle, be made from Eq. (4) if V_Σ were set equal to zero[12] and if only second-order diagrams of the form of Fig. 1(a) were used to calculate $V_{\Lambda\Sigma}$ and $V_{\Sigma\Lambda}$ and, finally, if the scattering amplitude were calculated in terms of the effective potential (4) in Born approximation.[14]

There is a more direct method, based on the Dyson S-matrix formalism, for determining the TPE contribution to the Λ-nucleon scattering amplitude (or scattering matrix) than that mentioned in the preceding paragraph. It is the purpose of this paper to report the results of a calculation of the TPE contribution to the Λ-nucleon scattering matrix in terms of the S matrix.

Breit[15] has recently discussed the TPE contribution to the nucleon-nucleon scattering amplitude in terms of the S matrix and its relation to a (fictitious) potential which reproduces this amplitude in Born approximation. He and his co-workers[16] have recently used such a potential, obtained by Gupta,[17] in an attempt to isolate TPE effects in nucleon-nucleon scattering. We have applied the S-matrix formalism to evaluate the TPE contribution to the Λ-nucleon scattering matrix. The method is outlined in Sec. II for the case of even $\Lambda-\Sigma$ parity and a universal pion-baryon interaction.[18] The results are given in Sec. III where it is shown that *the TPE contribution to the zero-energy scattering amplitude is spin-independent* and that there is a momentum-dependent spin dependence. Moreover, for Λ-nucleon scattering at finite energies, the TPE contribution to the Λ-nucleon scattering matrix contains an *antisymmetric spin-orbit term* of the form[19]

$$[\boldsymbol{\sigma}(1)-\boldsymbol{\sigma}(2)]\cdot\hat{n}, \tag{5}$$

where $\boldsymbol{\sigma}(1)$ and $\boldsymbol{\sigma}(2)$ are the Pauli spin operators of the two baryons and $\hat{n}$ is the unit normal to the scattering plane. The final Sec. IV contains a discussion of the results, including the suggestion that the low-energy Λ-nucleon potential should have a predominantly spin-independent tail.

[14] In the calculations reported in references 4 and 7, contributions to fourth-order Tamm-Dancoff potentials corresponding to diagrams with baryon-antibaryon pairs in intermediate states are omitted. These intermediate states are included in the S-matrix calculations reported in the present paper. For a discussion of the effects of such intermediate states see, for example, reference 10.

[15] G. Breit, Ann. Phys. (N. Y.) **16**, 346 (1961).

[16] G. Breit, K. E. Lassila, H. M. Ruppel, and M. H. Hull, Jr., Phys. Rev. Letters **6**, 138 (1961). See also G. Breit, Rev. Mod. Phys. **34**, 766 (1962).

[17] S. N. Gupta, Phys. Rev. **117**, 1146 (1960).

[18] Evidence that the $\Lambda-\Sigma$ parity is, in fact, even is given by R. D. Tripp, M. B. Watson, and M. Ferro-Luzzi, Phys. Rev. Letters **8**, 175 (1962).

[19] The possible existence and measurable consequences of an antisymmetric spin-orbit term were discussed by B. W. Downs and R. Schrils, Phys. Rev. **127**, 1388 (1962).

II. TPE CONTRIBUTION TO THE Λ-NUCLEON SCATTERING MATRIX

In this section, the connection between the scattering operator S and the scattering matrix M is established; and the TPE contribution to S for Λ-nucleon scattering is obtained.

The scattering operator for elastic Λ-nucleon scattering can be expressed in the form[20]

$$\begin{aligned} S-1=[i(2\pi)^2\hbar c]^{-1}\int d^3p''\int d^3q''\int d^3p'\int d^3q' \\ \times\delta^4(p'+q'-p''-q'')\psi_N^{\dagger(-)}(\mathbf{p}')\psi_\Lambda^{\dagger(-)}(\mathbf{q}') \\ \times\mathfrak{B}(\mathbf{p}',\mathbf{q}';\mathbf{p}'',\mathbf{q}'')\psi_N^{(+)}(\mathbf{p}'')\psi_\Lambda^{(+)}(\mathbf{q}''). \end{aligned} \tag{6}$$

The baryon field operators $\psi(x)=\psi^{(+)}(x)+\psi^{(-)}(x)$ have been represented by the Fourier integrals[21]

$$\psi^{(\pm)}(x)=(2\pi)^{-3/2}\int d^3p\,\psi^{(\pm)}(\mathbf{p})e^{(\mp)ip\cdot x}$$

in terms of the four-component momentum-space field operators $\psi^{(\pm)}(\mathbf{p})$ with $\psi^{\dagger(-)}(\mathbf{p})=[\psi^{(+)}(\mathbf{p})]^\dagger$, $\psi^{(+)}(\mathbf{p})$ containing annihilation operators. The operator $\mathfrak{B}$, which appears in Eq. (6), has been discussed in detail by Breit,[15] who has associated it with the Fourier transform of a fictitious potential which, in Born approximation, reproduces the effect of $(S-1)$.[22] This operator is introduced at this point in preparation for a suggestion (which is made in Sec. IV) that the fictitious potential, to which the TPE contribution to $\mathfrak{B}$ is related, be used to represent the Λ-nucleon interaction for large separations.

Since the scattering matrix M is usually defined in terms of Pauli spin operators,[23] it is convenient for a calculation of M to introduce a second operator $\boldsymbol{\mathfrak{B}}$ defined by[15]

$$\begin{aligned} \psi_N^{\dagger(-)}(\mathbf{p}')\psi_\Lambda^{\dagger(-)}(\mathbf{q}')\mathfrak{B}\psi_N^{(+)}(\mathbf{p}'')\psi_\Lambda^{(+)}(\mathbf{q}'') \\ =\psi_{NL}^{\dagger(-)}(\mathbf{p}')\psi_{\Lambda L}^{\dagger(-)}(\mathbf{q}')\boldsymbol{\mathfrak{B}}\psi_{NL}^{(+)}(\mathbf{p}'')\psi_{\Lambda L}^{(+)}(\mathbf{q}''), \end{aligned} \tag{7}$$

where $\psi_{NL}^{(+)}(\mathbf{p}'')$ is the large (two component) part of $\psi_N^{(+)}(\mathbf{p}'')$, etc. Equation (6) leads to an expression for the Λ-nucleon scattering amplitude in terms of matrix elements of the operator $\mathfrak{B}$.[15] With the definition (7), the amplitude in the zero-momentum frame for the elastic scattering of a nucleon of momentum $\mathbf{p}$ in spin

[20] Equation (6) is essentially Eq. (7) of reference 15, there being unimportant differences only in the normalization and designation of the baryon field operators.

[21] Four vectors are printed in lightface type; and the metric which is used in this paper is such that $p\cdot x=p_0x_0-\mathbf{p}\cdot\mathbf{x}$. The units of four momenta used here are those of a reciprocal length; see, for example, Eqs. (9).

[22] See also S. N. Gupta, Nucl. Phys. **24**, 160 (1961).

[23] See, for example, L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 847 (1952); R. H. Dalitz, Proc. Phys. Soc. (London) **A65**, 175 (1952); and C. R. Schumacher and H. A. Bethe, Phys. Rev. **121**, 1534 (1961).

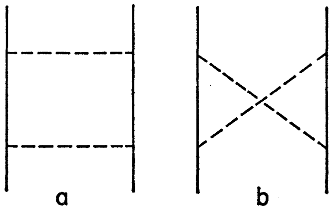

FIG. 3. TPE diagrams which contribute to S_4 for a Λ-nucleon system. One vertical line of each diagram is a nucleon line; the other, a hyperon (Λ and Σ) line.

state s_N by a Λ-particle of momentum $-\mathbf{p}$ in spin state s_Λ can be written

$$f(\theta,\varphi)=-(2\pi\hbar c)^{-1}[p_0q_0/(p_0+q_0)][(p_0+K_N)/2p_0]$$
$$\times[(q_0+K_\Lambda)/2q_0]\sum_{s_N'=1}^{2}\sum_{s_\Lambda'=1}^{2}\chi_{s_N'}\chi_{s_\Lambda'}$$
$$\times\langle\chi_{s_N'}\chi_{s_\Lambda'}|\mathfrak{B}(\mathbf{p}',-\mathbf{p}';\mathbf{p},-\mathbf{p})|\chi_{s_N}\chi_{s_\Lambda}\rangle, \quad (8)$$

in terms of Pauli (two component) spin functions χ for the two particles. The sums in (8) extend over final particle spin states, (θ,φ) are the polar angles of the final relative propagation vector $\mathbf{p}'$ with respect to the initial relative propagation vector $\mathbf{p}$; and

$$p_0=(K_N^2+\mathbf{p}^2)^{1/2}, \quad (9a)$$

$$q_0=(K_\Lambda^2+\mathbf{p}^2)^{1/2}, \quad (9b)$$

with

$$K_i=M_ic/\hbar. \quad (9c)$$

The elements of the M matrix are related to the scattering amplitude (8) by

$$M_{fi}=\langle(\chi_{s_N'}\chi_{s_\Lambda'})_f|f(\theta,\varphi)\rangle. \quad (10)$$

From (8) and (10) it follows that

$$M=-(2\pi\hbar c)^{-1}[p_0q_0/(p_0+q_0)][(p_0+K_N)/2p_0]$$
$$\times[(q_0+K_\Lambda)/2q_0]\mathfrak{B}. \quad (11)$$

The TPE contribution S_4 to $(S-1)$ corresponds to the diagrams of Fig. 3.[14] The $NN\pi$ interaction is taken to be the usual pseudoscalar-pseudoscalar interaction with coupling constant g_N. The corresponding $\Lambda\Sigma\pi$ interaction is taken to be[4,6]

$$g_{\Lambda\Sigma}\bar{\psi}_\Lambda\gamma_5\boldsymbol{\psi}_\Sigma\cdot\boldsymbol{\phi}_\pi+\text{H. c.}, \quad (12)$$

appropriate to the assumed even Λ−Σ parity. The TPE contribution to S_4 corresponding to the diagrams of Fig. 3 has been evaluated by Gupta for a system of two nucleons.[24] By a similar calculation, we find for the Λ-nucleon system[25]

$$S_4=[i3\pi^2g_N^2g_{\Lambda\Sigma}^2/(2\pi)^6\hbar^2c^2]$$
$$\times\int d^3p''\int d^3q''\int d^3p'\int d^3q'$$
$$\times\delta^4(p'+q'-p''-q'')$$
$$\times\sum_{\mu,\nu}\{[\bar{\psi}_N^{(-)}(\mathbf{p}')\gamma^\mu\psi_N^{(+)}(\mathbf{p}'')]$$
$$\times[\bar{\psi}_\Lambda^{(-)}(\mathbf{q}')\gamma^\nu\psi_\Lambda^{(+)}(\mathbf{q}'')]\}$$
$$\times\int_0^1du\int_0^u dv\int_0^v dw\{\tfrac{1}{2}g_{\mu\nu}[1/a^2-1/b^2]$$
$$-[h_\mu h_\nu-(\alpha-1)h_\mu q_\nu'']/(a^2)^2$$
$$+[f_\mu f_\nu-(\alpha-1)f_\mu q_\nu'']/(b^2)^2\}, \quad (13)$$

with

$$\alpha=K_\Sigma/K_\Lambda, \quad (14)$$

$$f=-wp''-(v-w)q'', \quad (15a)$$

$$h=wp''-(v-w)q''; \quad (15b)$$

and

$$a^2=(K_\Sigma^2-K_\Lambda^2)(v-w)+(K_\Lambda^2-K_N^2)(v-w)(v-2w)$$
$$+K_N^2(v-2w)^2+K_\pi^2(1-v)$$
$$-(p'-p'')^2(1-u)(u-v)-(p''-q'')^2(w-v)w$$
$$-[(p'-p'')^2+(p'-q'')^2-(p''-q'')^2]$$
$$\times(u-v)(v-w), \quad (16a)$$

$$b^2=(K_\Sigma^2-K_\Lambda^2)(v-w)+(K_\Lambda^2-K_N^2)(v-w)v$$
$$+K_N^2v^2+K_\pi^2(1-v)-(p'-p'')^2$$
$$\times[(1-u)(u-v)-(v-w)w]-(p''-q'')^2(v-w)w$$
$$-[(p'-p'')^2+(p'-q'')^2-(p''-q'')^2]$$
$$\times(u-v+w)(v-w), \quad (16b)$$

with K_i defined in (9c).

The TPE contribution to the Λ-nucleon scattering matrix M can be obtained from Eqs. (6), (7), (11), and (13) for the case of a universal pion-baryon interaction implied by (12).

III. THE FORM OF THE TPE M MATRIX

For the interaction of two spin-$\frac{1}{2}$ particles, the most general form of the scattering matrix, which is invariant under space rotations, space reflections, and time reversal, is[23]

$$M=A+B\sigma_n(1)\sigma_n(2)+C[\sigma_n(1)+\sigma_n(2)]$$
$$+D[\sigma_n(1)-\sigma_n(2)]+E\sigma_d(1)\sigma_d(2)+F\sigma_s(1)\sigma_s(2), \quad (17)$$

where $\sigma_n(i)$ is the component of $\boldsymbol{\sigma}(i)$ in the $\mathbf{n}$ direction, etc.; and $\mathbf{n}=\mathbf{p}\times\mathbf{p}'$ (normal to the scattering plane), $\mathbf{s}=\mathbf{p}'+\mathbf{p}$, and $\mathbf{d}=\mathbf{p}'-\mathbf{p}$ are three mutually perpendicu-

[24] The notation of Eqs. (12) and (13) is the usual one that a superscript bar indicates Hermitian adjoint multiplied from the right by γ^0. In reference 17, Gupta also considered fourth-order diagrams, other than those of Fig. 3, which do not contribute to S_4 for a Λ-nucleon system.

[25] The terms in (13) which contain a^2 arise from the diagram of Fig. 3(a); those which contain b^2, from that of Fig. 3(b).

lar vectors, $\mathbf{p}$ and $\mathbf{p}'$ being the initial and final relative momenta in the zero-momentum frame. The coefficients $A\cdots F$ are complex functions of the energy and scattering angles. Although the antisymmetric spin-orbit term $D[\sigma_n(1)-\sigma_n(2)]$ does not appear in the charge-symmetric nucleon-nucleon scattering matrix, it cannot be excluded from the Λ-nucleon scattering matrix on general grounds.[19]

Following the procedure outlined in the preceding section, we obtained the following expressions for the TPE contribution to the coefficients of the M matrix [with the sign of D determined by the choice $1=N$ and $2=\Lambda$ in Eq. (17)]:

$$\begin{aligned} A/G=I_2+x^2\{&[2I_1+I_3-I_5] \\ &+[\mathbf{p}'\cdot\mathbf{p}/\mathbf{p}^2][2I_1+I_2^{(+)}+I_3-I_5]\} \\ &+x^4\{I_7+[\mathbf{p}'\cdot\mathbf{p}/\mathbf{p}^2][2I_7+I_4-I_6] \\ &+[\mathbf{p}'\cdot\mathbf{p}/\mathbf{p}^2]^2[I_2+I_7+I_4-I_6]\}, \end{aligned} \tag{18a}$$

$$\begin{aligned} B/G=x^2\{&-2I_1+[\mathbf{p}'\cdot\mathbf{p}/\mathbf{p}^2][2I_1]\} \\ &+x^4\{[|\mathbf{p}\times\mathbf{p}'|/\mathbf{p}^2]^2[-I_2-I_7-I_4+I_6]\}, \end{aligned} \tag{18b}$$

$$\begin{aligned} C/iG=x^2\{&[|\mathbf{p}\times\mathbf{p}'|/\mathbf{p}^2][-2I_1-\tfrac{1}{2}I_2^{(+)}-\tfrac{1}{2}I_3+\tfrac{1}{2}I_5]\} \\ &+x^4\{[|\mathbf{p}\times\mathbf{p}'|/\mathbf{p}^2][-I_7-\tfrac{1}{2}I_4+\tfrac{1}{2}I_6] \\ &+[|\mathbf{p}\times\mathbf{p}'|/\mathbf{p}^2][\mathbf{p}'\cdot\mathbf{p}/\mathbf{p}^2] \\ &\times[-I_2-I_7-I_4+I_6]\}, \end{aligned} \tag{18c}$$

$$\begin{aligned} D/iG=x^2\{&[|\mathbf{p}\times\mathbf{p}'|/\mathbf{p}^2][-\tfrac{1}{2}I_2^{(-)}-\tfrac{1}{2}I_3-\tfrac{1}{2}I_5]\} \\ &+x^4\{[|\mathbf{p}\times\mathbf{p}'|/\mathbf{p}^2][\tfrac{1}{2}I_4+\tfrac{1}{2}I_6]\}, \end{aligned} \tag{18d}$$

$$E=0, \tag{18e}$$

$$F/G=x^2\{-2I_1+[\mathbf{p}'\cdot\mathbf{p}/\mathbf{p}^2][2I_1]\}, \tag{18f}$$

where

$$\begin{aligned} G=(3/2\pi)(g_N^2/4\pi\hbar c)(g_{\Lambda\Sigma}^2/4\pi\hbar c)[p_0q_0/(p_0+q_0)] \\ \times[(p_0+K_N)/2p_0][(q_0+K_\Lambda)/2q_0], \end{aligned} \tag{19a}$$

and

$$x^2=\mathbf{p}^2/(p_0+K_N)(q_0+K_\Lambda). \tag{19b}$$

All integrals I_i in the expressions (18) are of the form

$$I_i=\int_0^1 du\int_0^u dv\int_0^v dw\{I_i\}, \tag{20a}$$

the integrands being

$$\{I_1\}=1/2a^2-1/2b^2, \tag{20b}$$

$$\begin{aligned} \{I_2\}&=\{I_2^{(\pm)}\}/[(q_0+K_\Lambda)/(p_0+K_N) \\ &\qquad\pm(p_0+K_N)/(q_0+K_\Lambda)] \\ &=\{I_1\}-([(q_0+p_0)w-q_0v]^2 \\ &\qquad-(\alpha-1)q_0[(q_0+p_0)w-q_0v])/(a^2)^2 \\ &\qquad+([(q_0-p_0)w-q_0v]^2 \\ &\qquad-(\alpha-1)q_0[(q_0-p_0)w-q_0v])/(b^2)^2, \end{aligned} \tag{20c}$$

$$\begin{aligned} &\{I_3\}/(q_0+K_\Lambda) \\ &=\{I_4\}/(p_0+K_N) \\ &=v([(q_0+p_0)w-q_0v]-(\alpha-1)q_0)/(a^2)^2 \\ &\quad-(v-2w)([(q_0-p_0)w-q_0v]-(\alpha-1)q_0)/(b^2)^2, \end{aligned} \tag{20d}$$

$$\begin{aligned} &\{I_5\}/(p_0+K_N) \\ &=\{I_6\}/(q_0+K_\Lambda) \\ &=[(q_0+p_0)w-q_0v][v+(\alpha-1)]/(a^2)^2 \\ &\quad-[(q_0-p_0)w-q_0v][(v-2w)+(\alpha-1)]/(b^2)^2, \end{aligned} \tag{20e}$$

$$\begin{aligned} &\{I_7\}/(p_0+K_N)(q_0+K_\Lambda) \\ &=[v^2+(\alpha-1)v]/(a^2)^2 \\ &\quad-[(v-2w)^2+(\alpha-1)(v-2w)]/(b^2)^2. \end{aligned} \tag{20f}$$

In the integrands (20), a^2 and b^2 are given by (16) with $\mathbf{p}''$ replaced by $\mathbf{p}$ and $\mathbf{q}''$ replaced by $-\mathbf{p}$; then, in the zero-momentum frame, the last term in both (16a) and (16b) vanishes. A part of the momentum dependence of the M-matrix coefficients has been isolated in the factors x^2 and x^4; the remainder is contained in p_0, q_0 and the integrals $I_1\cdots I_7$.

In the static limit ($x^2=0$), the only nonvanishing scattering-matrix coefficient in (17) is A; that is,

$$M(x^2=0)=[GI_2]_{\mathbf{p}^2=0} \tag{21}$$

gives a TPE M matrix in the static limit, which is spin-independent. The TPE mechanism with the universal pion-baryon interaction assumed here cannot lead to the spin-dependent zero-energy scattering lengths deduced from analyses of hypernuclear binding-energy data.[1]

For Λ-nucleon scattering at finite energies, the TPE M matrix is spin-dependent. The relative importance of the spin-dependent terms in M, for a given value of the relative momentum, is difficult to estimate without an explicit calculation because of the rather complicated momentum dependence of the integrals I_i. The coefficients of the spin-dependent terms can, however, easily be obtained for small values of $(\mathbf{p}^2/K_NK_\Lambda)$ from the terms proportional to x^2 in Eqs. (18b)–(18f) by setting $p_0=K_N$, $q_0=K_\Lambda$, and evaluating the integrals I_i for $p^2=0$. We have evaluated these integrals numerically with the following results, expressed in units of $(3/2\pi)(g_N^2/4\pi\hbar c)(g_{\Lambda\Sigma}^2/4\pi\hbar c)\times10^{-2}$ F[26]:

$$B\underset{x^2\to0}{\longrightarrow}2.9_1[-1+(\mathbf{p}'\cdot\mathbf{p}/\mathbf{p}^2)][\mathbf{p}^2/K_NK_\Lambda], \tag{22a}$$

$$C\underset{x^2\to0}{\longrightarrow}-1.7_7i[|\mathbf{p}\times\mathbf{p}'|/\mathbf{p}^2][\mathbf{p}^2/K_NK_\Lambda], \tag{22b}$$

$$D\underset{x^2\to0}{\longrightarrow}0.3_9i[|\mathbf{p}\times\mathbf{p}'|/\mathbf{p}^2][\mathbf{p}^2/K_NK_\Lambda], \tag{22c}$$

$$F\underset{x^2\to0}{\longrightarrow}2.9_1[-1+(\mathbf{p}'\cdot\mathbf{p}/\mathbf{p}^2)][\mathbf{p}^2/K_NK_\Lambda]. \tag{22d}$$

For comparison, the static value of the coefficient A is, in the same units,[27]

$$A(x^2=0)=8.2_0. \tag{22e}$$

[26] We obtained the following static values for the basic integrals in units of 10^{-2} F^2: $I_1=2.2_5$, $I_2=3.1_8$, $I_3=-6.1_4$, and $I_5=3.8_5$.

[27] A self-consistent calculation of the TPE M matrix through order $(\mathbf{p}^2/K_NK_\Lambda)$ would require an expansion of the terms proportional to x^0 and x^2 in (18a) in terms of this ratio.

Since the occurrence of an antisymmetric spin-orbit term in the M matrix marks a qualitative difference between the Λ-nucleon and nucleon-nucleon interactions, it is of interest to note that Eqs. (22b) and (22c) indicate that the contribution of the antisymmetric term to the M matrix can be expected to be comparable to that of the symmetric spin-orbit term. In this connection, it should be noted that neither the symmetric nor the antisymmetric spin-orbit terms in the TPE M matrix obtained in this paper will lead to detectable polarization effects because C and D are purely imaginary and A and B are purely real.[19,23] The calculations reported here do not, therefore, enable us to estimate the possible results of polarization experiments which we previously suggested for the determination of the existence of the antisymmetric spin-orbit term.[19] The feasibility of these experiments depends upon the addition of an imaginary part to A and/or B and/or the addition of a real part to D, arising from exchange mechanisms not considered here.

Finally, it is noted that, if the mass differences among the baryons are neglected, the antisymmetric spin-orbit term does not appear in M; the integrals in (18d) vanish in this case. This is to be expected because, with the neglect of these mass differences, the universal pion-baryon interaction assumed here implies that the Λ-nucleon interaction is directly related to the nucleon-nucleon interaction,[3,4,6] which does not lead to such an antisymmetric term.

IV. CONCLUDING REMARKS

The one-pion-exchange (OPE) M matrix has been found to lead to phase parameters which give a good representation of nucleon-nucleon scattering data for low energy and large values of angular momentum.[28] This leads one to expect that, for corresponding combinations of energy and angular momentum, the contribution of the TPE M matrix to Λ-nucleon scattering could be similarly isolated.[29] This follows from the expectation that the Λ-nucleon interaction corresponding to the TPE mechanism will be dominant for large separations just as OPE is expected to be dominant for large separations in the nucleon-nucleon interaction.[30] When sufficient Λ-nucleon scattering data have been accumulated to justify an analysis of the magnitude of that currently being pursued in the nucleon-nucleon case,[28] a calculation of the appropriate Λ-nucleon phase parameters from the TPE M matrix can presumably provide a test of assumed pion-hyperon interactions and a determination of the corresponding coupling constants.

For analyses of bound-state data and for simple analyses of scattering data, it is certainly convenient to have a representation of the Λ-nucleon interaction in terms of a local potential. It is, therefore, of interest to inquire to what extent the calculations of the TPE M matrix reported here can be used to indicate the form of an appropriate Λ-nucleon potential. The remainder of this final section is devoted to such an inquiry based on analogy with the corresponding situation in the nucleon-nucleon interaction.

The one-pion-exchange potential (OPEP) is that potential which, in Born approximation, reproduces the OPE M matrix.[31] When the nucleon-nucleon interaction has been represented by OPEP for large separations, it has led to qualitative agreement with those observed properties of the deuteron which are most sensitive to the form of the potential in the asymptotic region.[32] The success of the OPE mechanism in reproducing relevant scattering and bound-state data has led to the specification of semiphenomenological nucleon-nucleon potentials which, in the region of large separations ($r\gtrsim 1.5$–$2K_\pi$), are dominated by OPEP.[33]

The expectation that TPE should dominate the Λ-nucleon interaction in the region of large separations leads one to expect that TPEP (which, in Born approximation, reproduces the TPE M matrix) should provide a good representation of the Λ-nucleon interaction in that region. The relation between a local TPEP and the TPE M matrix is not as clear-cut as that between OPEP and the OPE M matrix on account of the rather complicated dependence of the TPE M matrix on $\mathbf{s}=\mathbf{p}'+\mathbf{p}$ in addition to its depencence on the momentum transfer $\mathbf{d}$ (and this $\mathbf{s}$ dependence would correspond to a TPEP with a nonlocal component). In the approximation in which the dependence of the TPE M matrix on $\mathbf{s}$ can be neglected, however, a local TPEP can be defined.[34] Such a local TPEP has been obtained for the nucleon-nucleon system by Gupta,[17] who evaluated only the dominant spin-independent part. Gupta's spin-independent TPEP has been used by Breit *et al.*[16] to improve agreement between empirical phase parameters and those calculated from OPEP in the region where the contribution of TPEP is expected to be important.

The remarks in the two preceding paragraphs suggest that an appropriate Λ-nucleon potential should have a form which, in the asymptotic region, is in agreement with the Λ-nucleon TPEP. This potential, correspond-

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[28] See, for example, reference 10; the review by M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Ann. Rev. Nucl. Sci. **10**, 291 (1960); A. F. Grashin, Zh. Eksperim. i Teor. Fiz. **36**, 1717 (1959) [translation: Soviet Phys.—JETP **9**, 1223 (1959)]; and P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959).

[29] The TPE contribution to nucleon-nucleon scattering has been investigated in reference 16.

[30] See the discussion in reference 15 and in the second reference in footnote 16.

[31] The restriction to Born approximation is made in order that higher order effects not be included by iterations; see, for example, the discussion in Sec. I of this paper. OPEP and TPEP are the second- and fourth-order fictitious potentials mentioned in connection with Eq. (6).

[32] See, for example, J. Iwadare, S. Otsuki, R. Tamagaki, and W. Watari, Progr. Theoret. Phys. (Kyoto) **16**, 455 (1956) and Suppl. Progr. Theoret. Phys. **3**, 32 (1956); and reference 10.

[33] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962); and K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald and G. Breit, Phys. Rev. **126**, 881 (1962).

[34] S. N. Gupta, Nuovo Cimento **18**, 823 (1960) and reference 17.

ing to the interaction (12), will, therefore, be predominantly spin-independent for large separations and low energies. In the approximation in which baryon mass differences are neglected,[35] the Λ-nucleon TPEP corresponding to the interaction (12) is just the TPEP obtained by Gupta[17] with $\boldsymbol{\tau}(1)\cdot\boldsymbol{\tau}(2)=0$ and $g_N{}^4=g_N{}^2g_{\Lambda\Sigma}{}^2$.[3,4,6] In the region of small separations, the form of the Λ-nucleon potential is presumably determined by meson-exchange mechanisms not considered here. The work reported here indicates that the spin dependence of the two-body Λ-nucleon interaction at low energies is primarily an attribute of the interior region of the potential, whose parameters will probably have to be determined phenomenologically.[36]

[35] Our calculations indicate that the neglect of mass differences may not be a very good approximation. For example, the static values of the integrals I_i (see footnote 26) are all of comparable magnitude, and they contribute to the M matrix coefficients in an additive manner; and yet I_3, I_4, I_5, and I_6 vanish when mass differences are neglected. Moreover, about one-third of the static value of I_2, which determines the M matrix for zero-energy scattering, arises from the terms in the integrand (20c) which are proportional to the $\Sigma-\Lambda$ mass difference $(\alpha-1)K_\Lambda$.

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[36] For a discussion of the possibility that a part of the spin-dependence of the Λ-nucleon interaction deduced in the references of footnote 1 can be attributed to three-body interactions see, for example, A. R. Bodmer and S. Sampanthar, Nucl. Phys. **31**, 251 (1962).

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New Series for Phase Shift in Potential Scattering

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A new series for the phase shift has been derived for the Schrödinger, Klein-Gordon, and Dirac equations. This series converges faster than the Born series for the tangent of the phase shift. This is so because the sum of the first n terms in the new series includes exactly all the terms up to the $2(2^n-1)$th order in the Born series. Under the condition which is tantamount to that the phase shift cannot be larger than 63°, the series converges absolutely. At high energies the series can be analytically continued with respect to the strength of the potential beyond such a limit. It is shown that the high-energy limit of the phase shift is given by its first Born approximation and that the difference between even and noneven potentials is reflected in the respective phase shifts to all orders.

1. INTRODUCTION

THE high-energy potential scattering has been studied extensively[1–7] using the Schrödinger, the Klein-Gordon, or the Dirac equation. Recently, the interest in high-energy potential scattering has been revived[8–10] with the hope that it may be possible to suggest something useful to the high-energy field theoretical scattering. It is the purpose of this paper to derive a new series for the phase shift and apply the new formula to high-energy scattering. This series gives a unique value for the phase shift based on its Born expansion. A result which carries important theoretical implications is obtained, but it is not our primary concern to improve on a practical method of computing a phase shift from a given potential.

It has been customary to start with a formula for the tangent of the phase shift or something equivalent to it. In this case the phase shift is determined only up to an arbitrary multiple of π. Therefore, two sets of phase shifts, which differ by an arbitrary step function of momentum whose value takes only some multiple of π, are equivalent to each other. Such an ambiguity cannot be removed when one uses the tangent of the phase shift.

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